Can heterogeneous preferences stabilize endogenous fluctuations?

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Can Heterogeneous Preferences Stabilize Endogenous Fluctuations?*

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Abstract

While most of the literature concerned with indeterminacy and endogenous cycles is based on the restrictive assumption of a representative consumer, some recent contributions have investigated the role of heterogeneous agents in dynamics. This paper adds to this latter strand of the literature by highlighting the effects of heterogeneity in consumers' preferences within an overlapping generations economy with capital accumulation, endogenous labor supply and consumption in both periods. Using a mean-preserving approach to heterogeneity, we show that increasing the dispersion of propensity to save decreases macroeconomic volatility, by narrowing down the range of parameter values compatible with indeterminacy and ruling out expectations-driven fluctuations under a sufficiently large heterogeneity.

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1 Introduction

In the last two decades, several papers have investigated conditions for indeterminacy and endogenous cycles in intertemporal general equilibrium models.[1] One of the criticisms of this approach concerns the assumption of a representative agent as the average behavior of either an infinitely-lived population or a

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[1] For a survey, the reader is referred to Benhabib and Farmer (1999).

finite-horizon generation. How the dynamic properties of the model depend on such a restrictive approach has been seldom explained.

Some recent papers have introduced heterogeneous infinitely-lived agents in dynamic models with capital accumulation. Ghiglino and Olszak-Duquenne (2001), Ghiglino (2005), Bosi and Seegmuller (2006) and Ghiglino and Venditti (2006) have focused on the role of consumers' diversity in the occurrence of optimal cycles, whereas Ghiglino and Sorger (2002) and Ghiglino and Olszak-Duquenne (2005) were mainly concerned with its influence on indeterminacy.[2] All these papers prove that heterogeneity matters for endogenous fluctuations, but no clear-cut results seem to emerge.

Similar conclusions hold in the overlapping generations literature. For instance, Ghiglino and Tvede (1995) study an exchange economy with many consumers and commodities.

In this paper, we connect the two previous types of contributions by analyzing an overlapping generations model with heterogeneous consumers and capital accumulation. The model describes a competitive economy, populated by consumers living for two periods, supplying labor when young and consuming during their whole life. As in d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1995), Seegmuller (2004) and Lloyd-Braga, Nourry and Venditti (2006), preferences are homogeneous and non-separable in current and future consumption, but separable in leisure. In this context, parameters of economic significance, such as the propensity to save or the elasticity of labor supply, emerge naturally.

While the dynamic consequences of heterogeneity in technologies or in endowments have been relatively more studied, we introduce heterogeneity through two types of consumers who differ only in their propensities to save. In order to assess only the effect of heterogeneity on the occurrence of endogenous fluctuations, we adopt a mean-preserving approach, while increasing consumers' dispersion. We also contribute to a recent debate on the existing link between indeterminacy and the degree of propensity to save in overlapping generations economies characterized by a representative consumer.[3]

The geometrical method developed by Grandmont, Pintus and de Vilder (1998) is convenient to analyze the effects of heterogeneity on local indeterminacy and local bifurcations. Under constant returns to scale, we find that increasing heterogeneity reduces the range of parameter values (elasticity of labor supply, capital-labor substitution) where indeterminacy occurs. In particular, fluctuations due to self-fulfilling prophecies no longer occur when the labor supply is sufficiently elastic. In addition, we prove that, beyond a threshold of saving rate dispersion, indeterminacy is definitely ruled out. We conclude that under constant returns, heterogeneity of preferences stabilizes endogenous

[2] In order to account for binding financing or borrowing constraints, some economists have introduced another kind of heterogeneity, through discount rates, in infinite-horizon models. See, among others, Woodford (1986), Becker and Foias (1987, 1994) and Sorger (1994).

[3] In this respect, Cazzavillan and Pintus (2004, 2006) show that indeterminacy requires a sufficiently high propensity to save in economies with constant returns to scale or capital externalities, whereas this is no longer the case with labor externalities (Lloyd-Braga, Nourry and Venditti (2006)).

fluctuations.

However, under constant returns and without heterogeneity, the emergence of endogenous fluctuations requires two quite restrictive conditions in overlapping generations economies: namely, weak substitution between capital and labor[4] and high propensity to save[5]. In their recent contribution, Lloyd-Braga, Nourry and Venditti (2006) prove that these restrictions are no longer needed in the presence of productive labor externalities. On the one side, (mild) labor externalities make the parameters range more plausible, and, on the other side, they allow us to check the robustness of results in economies characterized by increasing returns, even if expectations-driven fluctuations require slightly different conditions.

Finally, we show that indeterminacy becomes less likely under a higher degree of heterogeneity, because the range of parameter values for indeterminacy shrinks. Moreover, just like under constant returns, indeterminacy no longer occurs for a sufficiently elastic labor supply and it is ruled out when the dispersion of the propensity to save becomes large enough. Therefore, in contrast to most of the existing results, we provide clear-cut conditions about the influence of heterogeneity on endogenous fluctuations.

The paper is organized as follows. In the next section, we present the model and define the intertemporal equilibrium. Section 3 is devoted to the existence of a steady state. In section 4, we analyze local dynamics assuming, at first, constant returns and, then, increasing returns. Concluding remarks are provided in section 5, while computational details are gathered in the Appendix.

2 The model

Heterogeneity in consumers' preferences is introduced in a discrete time overlapping generations model with capital accumulation ($t = 1, 2, \ldots$). Markets are supposed to be perfectly competitive.

In contrast to the consumers' side, the production sector is homogeneous: A representative firm is supposed to produce a unique final good by means of a constant returns to scale technology which employs capital (k_{t-1}) and labor (l_t). Production is also affected by aggregate labor externalities $\psi(l)$.[6] The amount of final good y_t, yielded in period t, is given by $y_t = A\psi(l_t) f(a_t) l_t$, where $A > 0$ is a scaling parameter. f and $a_t \equiv k_{t-1}/l_t$ represent an intensive production function and the capital intensity, respectively. On the technological side, we further assume:

Assumption 1 *The production function $f(a)$ is continuous for $a \geq 0$, positive-valued and continuously differentiable as many times as needed for $a > 0$, with*

[4] This assumption is not in keeping with some recent empirical studies. See, in particular, Duffy and Papageorgiou (2000).

[5] This condition has been criticized by, among others, Cazzavillan and Pintus (2004).

[6] Capital externalities are not introduced in the economy because, as it is shown by Cazzavillan and Pintus (2006) and Lloyd-Braga, Nourry and Venditti (2006), they fail to promote endogenous fluctuations in overlapping generations models with consumption in both periods.

$f''(a) < 0 < f'(a)$.

The externality function $\psi(l)$ *is continuous for* $l \geq 0$*, positive-valued and differentiable as many times as needed for* $l > 0$*. Moreover,* $\varepsilon_\psi(l) \equiv l\psi'(l)/\psi(l) \geq 0$*.*

Notice that when $\varepsilon_\psi(l) = 0$, there are no externalities and returns to scale become constant, whereas under $\varepsilon_\psi(l) > 0$, returns to scale are increasing to a degree $1 + \varepsilon_\psi(l)$.

As usual, the representative firm maximizes the profit, which determines the real interest rate r_t and the real wage w_t. If we set $\rho(a) \equiv f'(a)$ and $\omega(a) \equiv f(a) - af'(a)$, we get immediately:

$$\begin{aligned} r_t &= A\psi(l_t)\rho(a_t) \equiv r(k_{t-1}, l_t) \\ w_t &= A\psi(l_t)\omega(a_t) \equiv w(k_{t-1}, l_t) \end{aligned}$$

Two identities of interest relate the elasticities of $\rho(a)$ and $\omega(a)$ to the capital share in total income $\alpha(a) \equiv af'(a)/f(a) \in (0,1)$ and to the elasticity of capital-labor substitution $\sigma(a) > 0$:[7]

$$a\rho'(a)/\rho(a) = -[1 - \alpha(a)]/\sigma(a) \tag{1}$$
$$a\omega'(a)/\omega(a) = \alpha(a)/\sigma(a) \tag{2}$$

On the consumption side, there are overlapping generations of two-period-lived consumers. Population is constant and the total size of a generation is normalized to unity. In order to keep things as simple as possible, but without loosing generality, we reduce consumers' heterogeneity to two types of agents, labeled with $i = 1, 2$. We note $\lambda_i \in [0,1]$ the relative size of the ith class of consumers, which is held constant over time. By definition, $\lambda_1 + \lambda_2 = 1$. Each agent supplies labor only in the first period of life, saves through productive capital and consumes in both periods.[8] Preferences of a consumer of type i are summarized by a non-separable utility function in consumption of both periods, but separable in consumption and labor:

$$U_i(c_{i1t}, c_{i2t+1})/B_i - v_i(l_{it}) \tag{3}$$

where c_{i1t} (c_{i2t+1}) is the consumption during the first (second) period of life, l_{it} the labor supply and $B_i > 0$ a scaling parameter.[9]

Assumption 2 *The function* $U_i(c_{i1}, c_{i2})$ *is continuous for* $c_{i1} \geq 0$ *and* $c_{i2} \geq 0$ *with continuous derivatives of any required order for* $c_{i1} > 0$ *and* $c_{i2} > 0$*. Moreover,* $U_i(c_{i1}, c_{i2})$ *is increasing in* c_{i1} *and* c_{i2}*, strictly quasi-concave, homogeneous of degree one and such that the underlying indifference curves never cross the axes.*

[7] Identities (1) and (2) are straightforwardly deduced from $1/\sigma(a) = a\omega'(a)/\omega(a) - a\rho'(a)/\rho(a)$ and $\omega'(a) = -a\rho'(a)$.

[8] The length of a period (half a life) accounts for the full capital depreciation during the period.

[9] Similar preferences have been used, among others, by d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1995), Seegmuller (2004), Lloyd-Braga, Nourry and Venditti (2006).

The function $v_i(l_i)$ is continuous for $0 \leq l_i \leq L_i$ with continuous derivatives of any required order for $0 < l_i < L_i$, where L_i denotes the positive, finite or even infinite, labor endowment. Furthermore, we assume that $v_i(l_i)$ is increasing and convex, and satisfies $\lim_{l_i \to 0} v_i'(l_i) = 0$ and $\lim_{l_i \to L_i} v_i'(l_i) = +\infty$.

In the youth, the labor income $w_t l_{it}$ is consumed (c_{i1t}) and saved (k_{it}). In the retirement age, the capital income $r_{t+1} k_{it}$ is consumed (c_{i2t+1}). So, the ith type consumer faces two budget constraints:

$$c_{i1t} + k_{it} = w_t l_{it} \tag{4}$$
$$c_{i2t+1} = r_{t+1} k_{it} \tag{5}$$

Consumers compute the optimal levels of consumption and saving by maximizing the utility function (3) under the budget constraints (4) and (5). Using the intertemporal condition:[10]

$$\frac{U_{i1}(c_{i1t}, c_{i2t+1})}{U_{i2}(c_{i1t}, c_{i2t+1})} = r_{t+1} \tag{6}$$

we find the optimal levels:

$$c_{i1t} = [1 - s_i(r_{t+1})] w_t l_{it} \tag{7}$$
$$c_{i2t+1} = r_{t+1} s_i(r_{t+1}) w_t l_{it} \tag{8}$$
$$k_{it} = s_i(r_{t+1}) w_t l_{it} \tag{9}$$

where $s_i(r_{t+1}) \in (0,1)$ is the propensity to save and $1 - s_i(r_{t+1})$ the propensity to consume (when young).[11] The consumption ratio becomes a function, say q_i, of the real interest rate

$$\frac{c_{i1t}}{c_{i2t+1}} = \frac{1 - s_i(r_{t+1})}{s_i(r_{t+1}) r_{t+1}} \equiv q_i(r_{t+1})$$

and the elasticity of intertemporal substitution $\eta_i > 0$ can be viewed as elasticity of q_i (in absolute value):

$$\eta_i(r_{t+1}) = -\frac{q_i'(r_{t+1}) r_{t+1}}{q_i(r_{t+1})} = 1 + \frac{s_i'(r_{t+1}) r_{t+1}}{s_i(r_{t+1})[1 - s_i(r_{t+1})]} \tag{10}$$

Of course, s_i is decreasing for $0 < \eta_i < 1$ (intertemporal complementarity), increasing for $\eta_i > 1$ (intertemporal substitutability) and constant for $\eta_i = 1$: Savings increase (decrease) with respect to the interest rate under intertemporal substitutability (complementarity), whereas they don't depend on r_{t+1} when $\eta_i = 1$.

Substituting (7) and (8) into the function $U_i(c_{i1t}, c_{i2t+1})$, we get the consumption utility level for a unit of labor income $w_t l_{it} = 1$: $U_i^*(r_{t+1}) / B_i$ with

[10] Where $U_{ij} \equiv \partial U_i(x_1, x_2) / \partial x_j$ denotes a marginal utility.
[11] More details are provided in the Appendix.

$U_i^* (r) \equiv U_i (1 - s_i (r), s_i (r) r)$. As shown in the Appendix, the saving rate is also the elasticity of U_i^*:

$$rU_i^{*\prime} (r) / U_i^* (r) = s_i (r) \in (0, 1)$$

By definition of U_i^*, the consumption-leisure arbitrage for a consumer of type i simplifies to:

$$U_i^* (r_{t+1}) w_t = B_i v_i' (l_{it})$$

Let $\varepsilon_{v_i} (l_i) \equiv l_i v_i'' (l_i) / v_i' (l_i) > 0$ be the elasticity of the marginal disutility of labor (Assumption 2): The labor supply increases in the real wage with elasticity $1/\varepsilon_{v_i} (l_i)$.

Since (aggregate) capital is given by $k_t = \lambda_1 k_{1t} + \lambda_2 k_{2t}$, where k_{it} is determined by (9), we can define an intertemporal equilibrium as a sequence $(k_{t-1}, l_t)_{t=1}^{\infty}$ that meets the following conditions:

$$\begin{aligned} k_t &= [\lambda_1 s_1 (r (k_t, l_{t+1})) l_{1t} + \lambda_2 s_2 (r (k_t, l_{t+1})) l_{2t}] w (k_{t-1}, l_t) && (11) \\ l_t &= \lambda_1 l_{1t} + \lambda_2 l_{2t} && (12) \end{aligned}$$

where the heterogeneous labor supplies are given by:

$$l_{it} \equiv v_i'^{-1} [U_i^* (r (k_t, l_{t+1})) w (k_{t-1}, l_t) / B_i] \equiv l_i (k_{t-1}, l_t, k_t, l_{t+1}) \qquad (13)$$

We remark that the capital k_{t-1} is the predetermined variable of this two-dimensional dynamic system (11)-(12). The intertemporal sequence of k_{t-1} and l_t, enables us to determine all the other variables, namely l_{it}, k_{it}, c_{i1t}, c_{i2t}, y_t.

In order to study the local dynamics and analyze the role of heterogeneity in preferences on the stability properties of the economy, we first establish the existence of a steady state and then we linearize the system in a neighborhood.

3 Steady state

A stationary state of dynamic system (11)-(12) is a solution (k, l) of the system:

$$\begin{aligned} k &= [\lambda_1 s_1 (r (k, l)) l_1 + \lambda_2 s_2 (r (k, l)) l_2] w (k, l) \\ l &= \lambda_1 l_1 + \lambda_2 l_2 \end{aligned}$$

with

$$l_i = v_i'^{-1} [U_i^* (r (k, l)) w (k, l) / B_i]$$

Following Cazzavillan, Lloyd-Braga and Pintus (1998), we prove the existence of a normalized steady state such that $k = l_1 = l_2 = 1$, that is $l = 1$, by setting appropriately the scaling parameters $A, B_1, B_2 > 0$.[12]

[12] For the sake of simplicity, we focus on local dynamics around the normalized steady state without characterizing the possible existence of other stationary states.

Proposition 1 *Under Assumptions 1 and 2, there exists a steady state of system (11)-(12) such that $k = l_1 = l_2 = 1$, and therefore $l = 1$, if and only if:*

$$\lim_{A \to +\infty} g(A) > 1/(\psi(1)[f(1) - f'(1)]) \tag{14}$$

where

$$g(A) \equiv A[\lambda_1 s_1(A\psi(1) f'(1)) + \lambda_2 s_2(A\psi(1) f'(1))] \tag{15}$$

and A, B_i are the unique solutions of:

$$\begin{aligned} g(A) &= 1/(\psi(1)[f(1) - f'(1)]) && (16) \\ B_i &= U_i^*(r(1,1))\, w(1,1)/v_i'(1) && (17) \end{aligned}$$

Proof. A steady state $k = l_1 = l_2 = 1$ is defined by (16)-(17). To establish the existence of this steady state, we need to prove that there is a unique solution $A > 0$, $B_i > 0$ to these equations. The function $g(A)$ defined by (15) is continuous and increasing (see the Appendix) and $\lim_{A \to 0} g(A) = 0$. Therefore, according to inequality (14), there is a unique solution $A > 0$ to equation (16). Moreover, since $U_i^*(r(1,1))\, w(1,1)/v_i'(1) > 0$, one can immediately see that there exist unique solutions $B_i > 0$ to (17). ■

Throughout the rest of the paper, Proposition 1 will be supposed to hold and no longer referred.

4 Local dynamics

In order to know how heterogeneity in consumers' preferences could affect the occurrence of endogenous fluctuations, we study the local dynamics. Two main findings deserve attention:

- An increase of heterogeneity in consumers' propensities to save stabilizes the economy by narrowing down the range of parameter values consistent with the multiplicity of equilibria, and hence with fluctuations due to self-fulfilling expectations;
- A sufficient degree of heterogeneity can definitely eliminate indeterminacy.

In order to characterize the local dynamics, we differentiate the system (11)-(12) in a neighborhood of the steady state $(k, l) = (1, 1)$ with $l_1 = l_2 = 1$. In what follows, we define $\alpha \equiv \alpha(1)$, $\sigma \equiv \sigma(1)$, $\varepsilon_\psi \equiv \varepsilon_\psi(1)$, $s_i \equiv s_i(A\psi(1) f'(1))$ and $\eta_i \equiv \eta_i(A\psi(1) f'(1))$, while ε_{v_i} denotes the elasticity of $v_i'(l_i)$ evaluated at the steady state.

Notice that propensities s_i, elasticities η_i and ε_{v_i} resume the fundamental information about preferences and, therefore, about consumers' heterogeneity.

For simplicity, we will focus on the case with no heterogeneity in the elasticities of labor disutility.[13] Finally, we also assume intertemporal substitutability between consumption in both periods for each type of agent.[14]

Assumption 3 $\varepsilon_{v_1} = \varepsilon_{v_2} \equiv \varepsilon_v$ *and* $\eta_i \geq 1$.

Denoting by J the Jacobian matrix evaluated at the steady state $(k, l) = (1, 1)$, local dynamics are represented by a linear system $(dk_t/k, dl_{t+1}/l)^T = J\,(dk_{t-1}/k, dl_t/l)^T$.

In the sequel, we exploit the fact that the trace T and the determinant D of the Jacobian matrix are the sum and the product of the eigenvalues, respectively. Following Grandmont, Pintus and de Vilder (1998), the stability properties of the system, that is, the location of the eigenvalues with respect to the unit circle, will be characterized in the (T, D)-plane (see Figures 1-6).[15] More precisely, we evaluate the characteristic polynomial $P(\mu) \equiv \mu^2 - T\mu + D = 0$ at -1, 0, 1. On the line (AB), one eigenvalue is equal to -1, i.e., $P(-1) = 1 + T + D = 0$. On the line (AC), one eigenvalue is equal to 1, i.e., $P(1) = 1 - T + D = 0$. On the segment $[BC]$, the two eigenvalues are complex conjugates with a unit modulus, i.e., $D = 1$ and $|T| < 2$. The steady state is a sink when $D < 1$ and $|T| < 1 + D$. It is a saddle point when $|1 + D| < |T|$. It is a source otherwise. Therefore, the steady state is locally indeterminate if and only if (T, D) is inside the triangle (ABC) and is locally determinate otherwise.[16] A transcritical bifurcation generically occurs when (T, D) crosses the line (AC), a flip bifurcation generically occurs when (T, D) crosses the line (AB), whereas a Hopf bifurcation generically emerges when (T, D) crosses the segment $[BC]$.

[13] Recently, Bosi and Seegmuller (2005) have characterized the role of heterogeneity in labor disutility on the occurrence of local indeterminacy in a closely related framework. Considering an overlapping generations model with consumption only in the second period of life, they have shown that consumers' preferences are summarized by a weighted elasticity of labor supply with respect to the real wage: In other terms, a mean-preserving increase of heterogeneity does not affect local dynamics.

[14] A similar assumption is made by Cazzavillan and Pintus (2004, 2006) and Lloyd-Braga, Nourry and Venditti (2006).

[15] Dynamic systems can be approximated and represented by a Jacobian matrix which in turn can be conveniently written in terms of elasticities and fundamental parameters. The geometrical approach focuses on some elasticity of economic significance and simplifies the bifurcation analysis whenever the image of the elasticity range is linear or simple-shaped in the (T, D)-plane. Moreover, few informations are needed to locate their intersections with the critical lines (AB) and (AC), and the critical segment $[BC]$.

[16] In our model, multiple equilibria converge to the sink (local indeterminacy), while a unique equilibrium converges to the saddle point (local determinacy). The source is also an equilibrium, if the initial capital corresponds to the stationary solution (but local determinacy has now measure zero).

The determinant D and the trace T of matrix J are given by:[17]

$$D = \frac{\alpha}{s}\frac{1+\varepsilon_v}{1-\alpha+\sigma\varepsilon_\psi} > 0 \tag{18}$$

$$T = \frac{\alpha + s(1-\alpha) + (s-\sigma-\varepsilon_s)\varepsilon_\psi + [\sigma + (1-\alpha)\varepsilon_s]\varepsilon_v + \left(1-\alpha-\frac{\varepsilon_\psi}{\varepsilon_v}\right)\frac{h}{s}}{s(1-\alpha+\sigma\varepsilon_\psi)} \tag{19}$$

where:

- $s \equiv \lambda_1 s_1 + \lambda_2 s_2$ is the average propensity to save weighted by the population sizes;
- $\varepsilon_s = \frac{\lambda_1 s_1}{\lambda_1 s_1+\lambda_2 s_2}\varepsilon_{s_1} + \frac{\lambda_2 s_2}{\lambda_1 s_1+\lambda_2 s_2}\varepsilon_{s_2}$ is the elasticity of the average saving rate s and can be reinterpreted as a weighted average of the individual elasticities $\varepsilon_{s_i} \equiv rs_i'(r)/s_i(r) = (1-s_i)(\eta_i - 1)$;
- $h \equiv \lambda_1 (s_1 - s)^2 + \lambda_2 (s_2 - s)^2$ is the variance of the propensities to save.

To understand the role of heterogeneity in preferences for local dynamics, we need to define a significant measure of heterogeneity and observe the consequences of raising this measure.

The propensity to save s_i is an informative parameter that captures the dynamic effect of preferences. As stressed by Cazzavillan and Pintus (2004, 2006) and Lloyd-Braga, Nourry and Venditti (2006), this parameter plays a key role on the occurrence of indeterminacy in overlapping generations economies with a representative consumer.[18]

To take one step forward, we address a worthwhile question: What are the implications on local indeterminacy when one raises the dispersion of saving rates in an economy with heterogeneous consumers? Since the most satisfactory way of appreciating the role heterogeneity is to keep the first order moments as given, we preserve explicitly the mean s of the propensities to save and the mean of their elasticities ε_s,[19] while raising their variance h, our significant measure of heterogeneity.

The main results of the paper are proved in the following: More heterogeneity in consumers' preferences reduces the range of parameters for indeterminacy and can rule out endogenous fluctuations. The cases of constant and increasing returns to scale are successively studied.

4.1 Constant returns ($\varepsilon_\psi = 0$)

Under constant returns to scale, the production sector does not benefit from labor externalities, *i.e.*, $\varepsilon_\psi = 0$. Local dynamics, that is, the occurrence of

[17] More details are provided in the Appendix.

[18] In order to obtain indeterminacy, the propensity to save is required to be sufficiently high under constant returns to scale or capital externalities, while such restriction is no longer needed in the presence of labor externalities.

[19] Note that keeping ε_s as constant is possible through an appropriate choice of η_1 and η_2.

indeterminacy and endogenous cycles, are characterized through an increase of heterogeneity in preferences.

According to empirical estimates, the capital share in total income α is supposed to be smaller than one half. In addition, we consider a sufficiently high average propensity to save s and we assume the elasticities of intertemporal substitution η_i to be sufficiently close to one.

Assumption 4 *$\alpha < 1/2$, $\varepsilon_s < \alpha / (1-\alpha) < s$.*

Consider the general expressions (18) and (19), and let

$$D_0 \equiv \alpha / [(1-\alpha) s] \tag{20}$$
$$T_0 \equiv 1 + D_0 + h/s^2 \tag{21}$$

be the determinant and the trace when the labor supply is infinitely elastic ($\varepsilon_v = 0$) under constant returns ($\varepsilon_\psi = 0$). Still using (18) and (19), the trace and the determinant can now be written as:

$$\begin{aligned} D &= D_0 (1 + \varepsilon_v) \\ T &= T_0 + D_0 \varepsilon_v / S(\sigma) \end{aligned}$$

where

$$S(\sigma) \equiv \alpha / [\sigma + (1-\alpha) \varepsilon_s] \tag{22}$$

When the bifurcation parameter ε_v varies from 0 to $+\infty$, the pair (T, D) describes a half-line Δ_0 in the (T, D)-plane, with origin (T_0, D_0) and slope $S(\sigma)$.

The origin (T_0, D_0) does not depend on σ, while D_0 and the slope S don't depend on h (see (20), (21) and (22)). The two parameters σ and h characterize unambiguously the position of Δ_0: When σ increases, Δ_0 makes a clockwise rotation around the invariant starting point (T_0, D_0), while Δ_0 moves on the right making a horizontal translation, when h increases.

More precisely, on the one hand, the slope $S(\sigma)$ decreases from $S(0) = \alpha / [(1-\alpha) \varepsilon_s]$, which is greater than 1 under Assumption 4, to 0, as long as σ tends to $+\infty$, and $S(\sigma) = 1$ for $\sigma = \alpha - (1-\alpha) \varepsilon_s \equiv \sigma_T$. On the other hand, T_0 increases with h, while D_0 remains invariant and belongs to $(0, 1)$ under Assumption 4.

According to (21), without heterogeneity ($h = 0$) the starting point (T_0, D_0) is on the line (AC) between the horizontal axis and C. Moreover, when $\sigma < \sigma_T$, the half-line Δ_0 lies above the line (AC) and crosses the segment $[BC]$ for $\varepsilon_v = s(1-\alpha)/\alpha - 1 \equiv \varepsilon_{v_H}$, whereas Δ_0 lies below the line (AC) for all $\sigma > \sigma_T$ (see Figure 1).

A positive degree of heterogeneity in the propensities to save, makes h strictly positive. The origin (T_0, D_0) turns out to lie below the line (AC).

First consider a slight degree of heterogeneity:

$$h < h_1 \equiv s \frac{1-\alpha}{\alpha} \left(s - \frac{\alpha}{1-\alpha} \right) \left(\frac{\alpha}{1-\alpha} - \varepsilon_s \right)$$

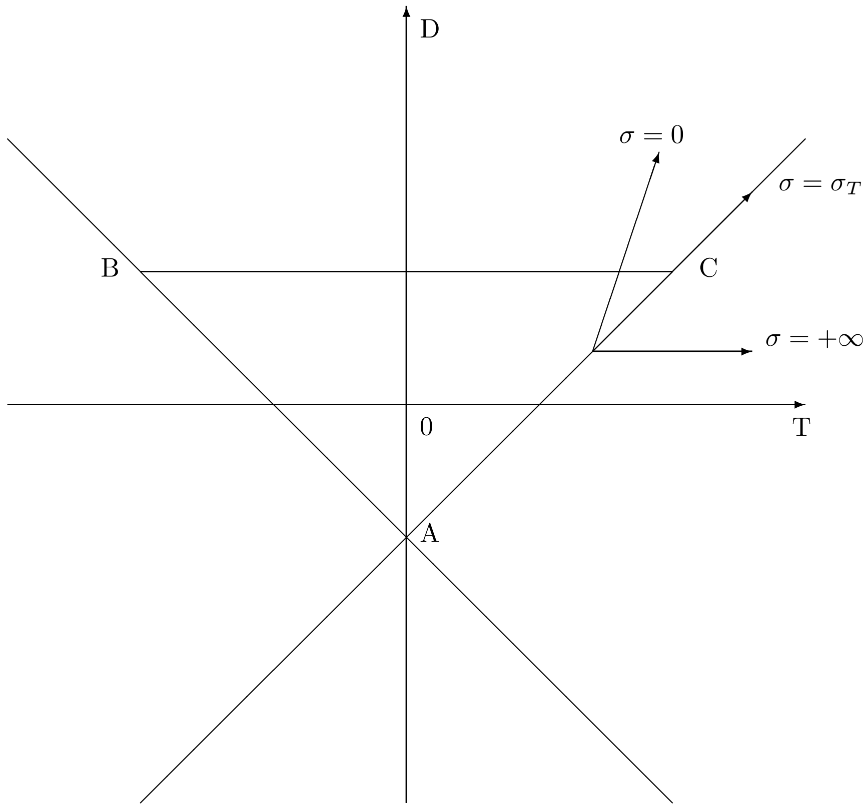

Figure 1: Constant returns ($\varepsilon_\psi = 0$) and no heterogeneity ($h = 0$)

and define $\varepsilon_{v_T} \equiv h\,(1-\alpha)\,/\,[s\,(\alpha-(1-\alpha)\,\varepsilon_s-\sigma)]$ as the bifurcation value for ε_v corresponding to the intersection of Δ_0 and (AC). We also note that the critical value of σ such that Δ_0 goes through C is given by:

$$\sigma_C \equiv \alpha-(1-\alpha)\,\varepsilon_s-\frac{1-\alpha}{s\,(1-\alpha)\,/\alpha-1}\frac{h}{s}\in(0,\sigma_T)$$

Therefore, when $\sigma < \sigma_C$, the half-line Δ_0, which starts below the line (AC), first crosses (AC) below C and then the segment $[BC]$. When $\sigma_C < \sigma < \sigma_T$, the slope $S\,(\sigma)$ remains greater than 1, but Δ_0 crosses now the line (AC) above C. Finally, when $\sigma > \sigma_T$, the slope $S\,(\sigma)$ becomes smaller than 1 and Δ_0 lies entirely below the line (AC) (see Figure 2).

Assume now a higher degree of heterogeneity $h \geq h_1$: The critical value σ_C becomes negative. This means that, whatever σ (even close to 0), the half-line Δ_0 lies outside the triangle (ABC), on the right side of C (see Figure 3). Therefore, when $\sigma < \sigma_T$, the half-line Δ_0, which starts below the line (AC), crosses (AC) above C. When $\sigma \geq \sigma_T$, as seen, Δ_0 lies entirely below (AC).

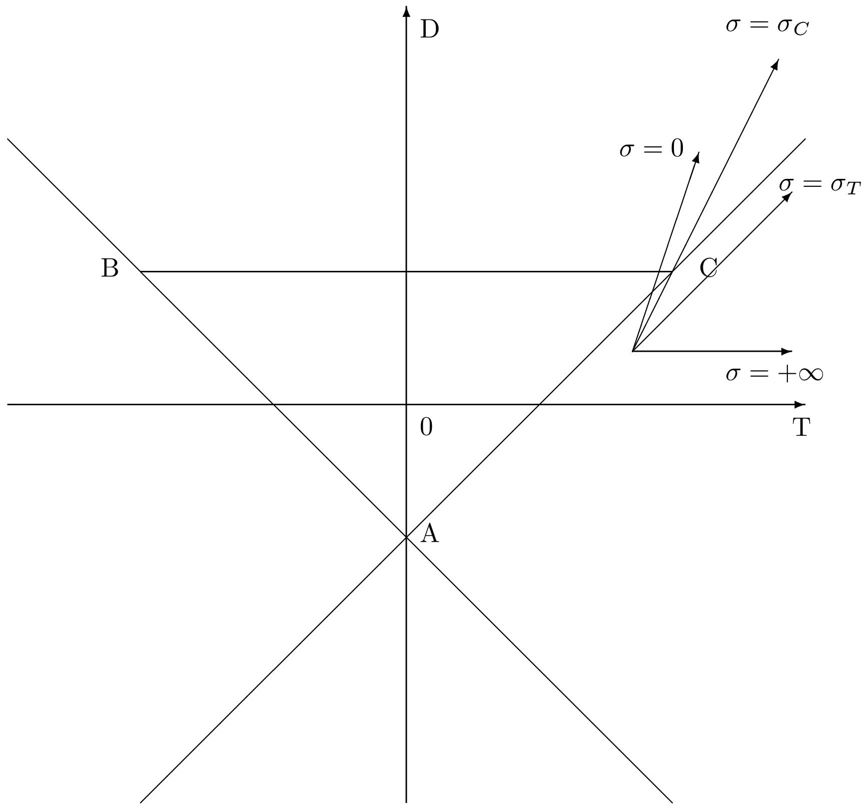

Figure 2: Constant returns ($\varepsilon_\psi = 0$) and heterogeneity with $h > 0$ not too large

These results are summarized in the next proposition:

Proposition 2 ***(Constant returns)*** *If Assumptions 3-4 are satisfied, the following generically holds.*

1. *No heterogeneity ($h = 0$).*

   (i) *When $0 < \sigma < \sigma_T$, the steady state is a sink for $0 < \varepsilon_v < \varepsilon_{v_H}$, undergoes a Hopf bifurcation at $\varepsilon_v = \varepsilon_{v_H}$, is a source for $\varepsilon_v > \varepsilon_{v_H}$.*

   (ii) *When $\sigma > \sigma_T$, the steady state is a saddle for all $\varepsilon_v > 0$.*

2. *Moderate heterogeneity ($0 < h < h_1$).*

   (i) *When $0 < \sigma < \sigma_C$, the steady state is a saddle for $0 < \varepsilon_v < \varepsilon_{v_T}$, undergoes a transcritical bifurcation at $\varepsilon_v = \varepsilon_{v_T}$, is a sink for $\varepsilon_{v_T} < \varepsilon_v < \varepsilon_{v_H}$, undergoes a Hopf bifurcation at $\varepsilon_v = \varepsilon_{v_H}$, is source for $\varepsilon_v > \varepsilon_{v_H}$.*

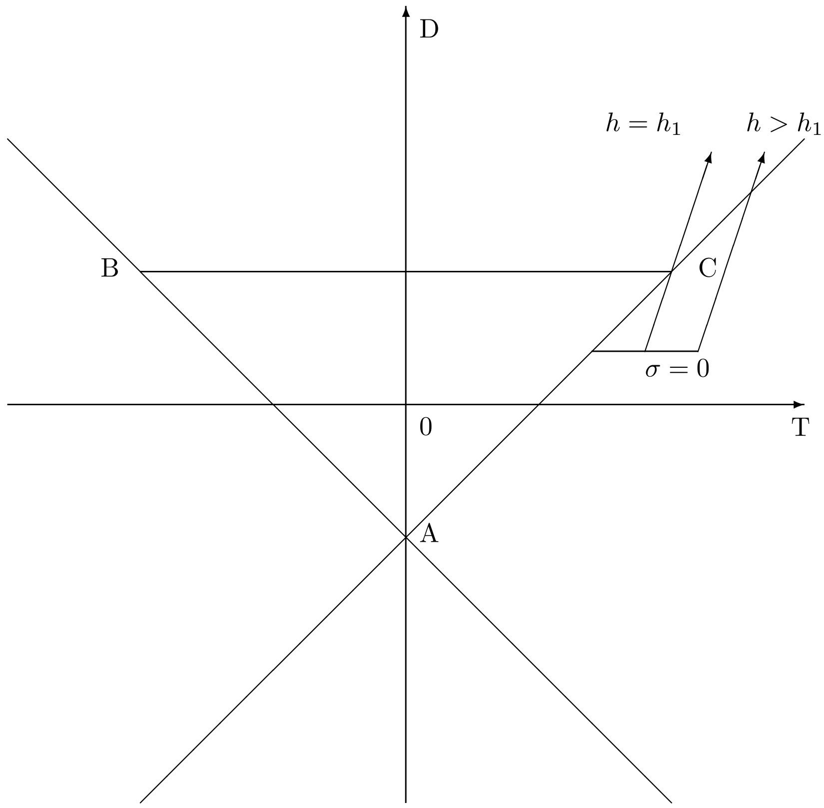

Figure 3: Constant returns ($\varepsilon_\psi = 0$) and heterogeneity with $h > h_1$

(ii) *When $\sigma_C < \sigma < \sigma_T$, the steady state is a saddle for $0 < \varepsilon_v < \varepsilon_{v_T}$, undergoes a transcritical bifurcation at $\varepsilon_v = \varepsilon_{v_T}$, is a source for $\varepsilon_v > \varepsilon_{v_T}$.*

(iii) *When $\sigma \geq \sigma_T$, the steady state is a saddle for all $\varepsilon_v > 0$.*

3. *Large heterogeneity ($h \geq h_1$).*

(i) *When $0 < \sigma < \sigma_T$, the steady state is a saddle for $0 < \varepsilon_v < \varepsilon_{v_T}$, undergoes a transcritical bifurcation at $\varepsilon_v = \varepsilon_{v_T}$, is a source for $\varepsilon_v > \varepsilon_{v_T}$.*

(ii) *When $\sigma \geq \sigma_T$, the steady state is a saddle for all $\varepsilon_v > 0$.*

Proposition 2 shows that without heterogeneity in the propensities to save (case 1), expectations-driven fluctuations need not only a weak capital-labor substitution ($\sigma < \sigma_T \leq \alpha$), but also a sufficiently elastic labor supply with respect to the real wage. In particular, indeterminacy can emerge under an infinitely elastic labor supply, that is, a linear disutility of labor ($\varepsilon_v = 0$).

Under a moderate degree of heterogeneity (case 2), on the one side, the emergence of endogenous fluctuations requires a weaker elasticity of substitution between capital and labor ($\sigma < \sigma_C < \sigma_T$). Furthermore, since σ_C linearly decreases with h, the larger the heterogeneity degree, the smaller the range of capital-labor substitution compatible with indeterminacy. On the other side, heterogeneity has also a negative effect on indeterminacy by reducing the range $(\varepsilon_{v_T}, \varepsilon_{v_H})$, given $\sigma < \sigma_C$. If the higher bound ε_{v_H} does not depend on the degree of heterogeneity,[20] there is now a lower bound $\varepsilon_{v_T} > 0$, that increases with the level of heterogeneity. This means that endogenous fluctuations are no longer possible under highly or even infinitely elastic labor supply, in sharp contrast with most of the existing results, mainly found using one-sector models, which suggest that a more elastic labor supply promotes fluctuations due to animal spirits.

Eventually, when heterogeneity becomes sufficiently high (case 3), the occurrence of endogenous fluctuations is ruled out. In fact, indeterminacy (and Hopf bifurcations as well) no longer occurs, when the second order moment h is higher than a threshold.[21]

We are now able to provide an interpretation of these results, by focusing on the existence of self-fulfilling expectations. For simplicity, we restrict our attention to the case where the propensities to save are constant, i.e., $\eta_1 = \eta_2 = 1$, so entailing $\varepsilon_s = 0$. Assume that agents coordinate their expectations on an increase of the future real interest rate. Since $dl_{it}/l_i = (s_i/\varepsilon_v)(dr_{t+1}/r)$, each agent increases his labor supply. Therefore, the effect on the aggregate labor supply is determined by $dl_t/l = \lambda_1 dl_{1t}/l_1 + \lambda_2 dl_{2t}/l_2 = (s/\varepsilon_v)(dr_{t+1}/r)$. Noticing

$$k_t = (\lambda_1 s_1 l_{1t} + \lambda_2 s_2 l_{2t})\, w_t$$

we observe that an increase of labor supply of each type of agent has two effects on capital accumulation, one through the individual labor supplies and their impacts on the terms in parentheses, another one through the aggregate labor supply and its impact on the wage. Taking into account these two channels, the variation of capital accumulation as a consequence of an increase in the expected real interest rate, depends on two opposite effects defined by:

1. $\frac{dk_t}{k} = \left(\frac{s}{\varepsilon_v} + \frac{h}{s\varepsilon_v}\right)\frac{dr_{t+1}}{r}$;

2. $\frac{dk_t}{k} = -\frac{\alpha}{\sigma}\frac{s}{\varepsilon_v}\frac{dr_{t+1}}{r}$.

Expectations are self-fulfilling if capital accumulation reduces, because, in this case, the future real interest rate increases. Therefore, the second (negative) effect has to dominate the first one. Without heterogeneity ($h = 0$), this requires $\sigma < \sigma_T = \alpha$. However, increasing the degree of heterogeneity ($h > 0$) reinforces the first effect which promotes determinacy. This explains why, when consumers

[20] The invariance of ε_{v_H} entails also that the instability range $(\varepsilon_{v_H}, +\infty)$ does not widen.

[21] We remark that, since $0 < s_i < 1$, h is strictly less than $1/4$ and thus the result established in the third case of Proposition 2 matters, if either s or ε_s remain close to the bound $\alpha/(1-\alpha)$, in order to ensure that $h_1 < 1/4$.

are heterogeneous, indeterminacy requires more restrictive conditions and can be even ruled out. One can further notice that, by direct inspection of the first effect, the smaller ε_v, the more stringent the impact of heterogeneity.

As we have seen, the occurrence of endogenous fluctuations under constant returns to scale requires at least two demanding conditions. On one hand, one need a sufficiently weak degree of substitution between capital and labor, which is not in accordance with empirical results (see Duffy and Papageorgiou (2000)). As shown in Lloyd-Braga, Nourry and Venditti (2006), this condition is no longer required as soon as labor externalities are introduced in the production sector. On the other hand, a too high propensity to save is open to criticism as well.[22] However, as stressed also by Lloyd-Braga, Nourry and Venditti (2006), under productive labor externalities, this assumption is no longer needed.

So there are at least two good reasons to study the case where returns to scale are increasing under the effect of labor externalities and to check the robustness of the findings obtained under constant returns.

4.2 Increasing returns ($\varepsilon_\psi > 0$)

Henceforth, we assume $\varepsilon_\psi > 0$, yielding positive externalities and increasing returns. In order to verify the robustness of our results, namely the positive role of heterogeneity in preferences for equilibrium determinacy, we study how local dynamics and the stability properties of the steady state vary in response to a mean-preserving change in h, the variance of saving rates.

To keep matters as simple as possible, not only we maintain Assumption 3, but also we assume an average propensity to save neither too low nor too high (as suggested by empirical evidence). In addition, according to the empirical literature, a too weak capital-labor substitution is excluded (see Duffy and Papageorgiou (2000)) and, as under constant returns to scale, a too large intertemporal substitution is not allowed.

Assumption 5 *Let* $\varepsilon_s < s < \alpha / (1-\alpha)$ *with* $s > 1 - \left(3 - \sqrt{1+8\alpha}\right) / \left[2(1-\alpha)\right]$, *and* $\sigma > \max\left\{\alpha - (1-\alpha)\varepsilon_s, (s-\varepsilon_s)/(1-s)\right\}$.

As above, we characterize the stability properties of the steady state and the occurrence of bifurcations in the (T, D)-plane and we choose $\varepsilon_v \in (0, +\infty)$ as bifurcation parameter to handle.

The analysis is made more difficult by the non-linear term $\varepsilon_\psi/\varepsilon_v$ appearing in (19). A direct inspection of (18) and (19) tells us that, as ε_v varies, (T, D) describes two different kind of curves for $h = 0$ and $h > 0$. In the first case, the locus is a half-line, whereas in the second one, it becomes a branch of hyperbole. For the sake of clarity, it is appropriate to study first local dynamics without heterogeneity and then to stress the main differences arising when h becomes strictly positive.

[22] See, among others, Cazzavillan and Pintus (2004).

4.2.1 Representative agent ($h = 0$)

Assuming $h = 0$, let

$$\begin{aligned} D_1 &= \alpha / \left[s \left(1 - \alpha + \sigma \varepsilon_\psi \right) \right] \\ T_1 &= 1 + D_1 \left(1 - \varepsilon_\psi \left[\sigma \left(1 + s \right) + \varepsilon_s - s \right] / \alpha \right) \end{aligned} \tag{23}$$

be the trace and the determinant when $\varepsilon_v = 0$ (infinitely elastic labor supply). Using (18) and (19), the determinant D and the trace T simplify:

$$D = D_1 \left(1 + \varepsilon_v \right) \tag{24}$$

$$T = T_1 + D_1 \varepsilon_v / S \left(\sigma \right) \tag{25}$$

where $S(\sigma)$ is still given by (22). When ε_v increases from 0 to $+\infty$, (T, D) describes a half-line Δ_1 with origin (T_1, D_1) and slope $S(\sigma)$, which is given by (22) and belongs to $(0, 1)$ under Assumption 5.

We compute two critical degrees of externality:

$$\begin{aligned} \varepsilon_{\psi_H} &\equiv \left[\alpha - s \left(1 - \alpha \right) \right] / \left(\sigma s \right) \\ \varepsilon_{\psi_F} &\equiv 2 \left[\alpha + s \left(1 - \alpha \right) \right] / \left[\sigma \left(1 - s \right) + \varepsilon_s - s \right] \end{aligned}$$

First we notice that Assumption 5 ensures $\varepsilon_{\psi_H} < \varepsilon_{\psi_F}$.[23] Moreover,

$$1 - T_1 + D_1 = \varepsilon_\psi \left[\sigma \left(1 + s \right) + \varepsilon_s - s \right] D_1 / \alpha \tag{26}$$

$$1 + T_1 + D_1 = \left(\varepsilon_{\psi_F} - \varepsilon_\psi \right) \left[\sigma \left(1 - s \right) + \varepsilon_s - s \right] D_1 / \alpha \tag{27}$$

In order to locate the origin of Δ_1, we find from (24) that D_1 belongs to $(0, 1)$ when $\varepsilon_{\psi_H} < \varepsilon_\psi$. The last inequality in Assumption 5 implies that the right-hand sides of (26) and (27) are positive if $\varepsilon_\psi > 0$ and $\varepsilon_\psi < \varepsilon_{\psi_F}$, respectively. So when $\varepsilon_{\psi_H} < \varepsilon_\psi < \varepsilon_{\psi_F}$, the starting point (T_1, D_1) lies above the horizontal axis inside (ABC) and, since T and D are both increasing in ε_v and the slope of Δ_1 belongs to $(0, 1)$, the half-line Δ_1 crosses the line (AC) or the segment $[BC]$.

In what follows, we define ε_{v_H} and ε_{v_T} the critical values of ε_v such that $D = 1$ and $1 - T + D = 0$, respectively. From (24)-(25), it follows that:

$$\begin{aligned} \varepsilon_{v_H} &\equiv s \left(1 - \alpha + \sigma \varepsilon_\psi \right) / \alpha - 1 = 1/D_1 - 1 \\ \varepsilon_{v_T} &\equiv \varepsilon_\psi \left[\sigma \left(1 + s \right) + \varepsilon_s - s \right] / \left[\sigma + \left(1 - \alpha \right) \varepsilon_s - \alpha \right] \end{aligned} \tag{28}$$

We notice that $\varepsilon_{v_H} < \varepsilon_{v_T}$ if and only if:

$$\varepsilon_\psi \left[s\sigma - \alpha \frac{\sigma \left(1 + s \right) + \varepsilon_s - s}{\sigma + \left(1 - \alpha \right) \varepsilon_s - \alpha} \right] < \alpha - s \left(1 - \alpha \right) \tag{29}$$

Under condition (29), the half-line lies above C, *i.e.*, Δ_1 crosses the segment $[BC]$ before the line (AC) (see Figure 4).[24]

[23] The inequality $\varepsilon_{\psi_H} < \varepsilon_{\psi_F}$ is equivalent to $(s - \varepsilon_s) \left[s \left(1 - \alpha \right) - \alpha \right] < \sigma \left[\left(1 - \alpha \right) s^2 + \left(1 + 2\alpha \right) s - \alpha \right]$. Under Assumption 5, the left-hand side is strictly negative, while the right-hand side is strictly positive.

[24] We observe that inequality (29) is compatible with $\varepsilon_{\psi_H} < \varepsilon_\psi$.

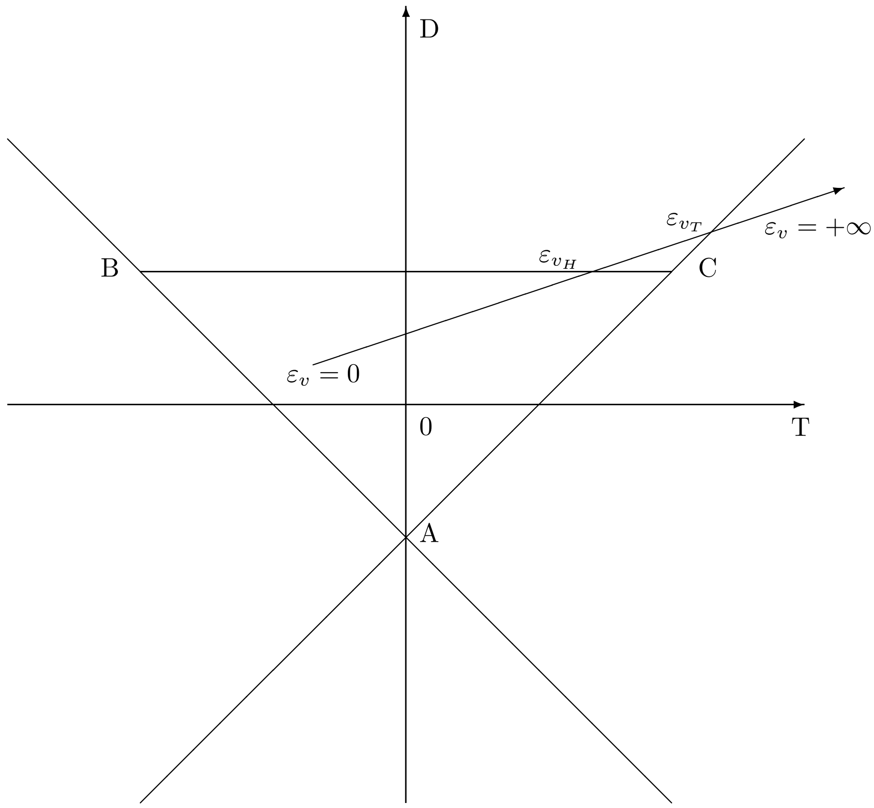

Figure 4: Increasing returns ($\varepsilon_\psi > 0$) and no heterogeneity ($h = 0$)

We can now summarize the conditions for indeterminacy and endogenous cycles when there is no heterogeneity in the propensities to save, *i.e.*, $h = 0$.

Proposition 3 ***(Increasing returns without heterogeneity*** $(h = 0)$***)*** *If Assumptions 3 and 5, and inequalities $\varepsilon_{\psi_H} < \varepsilon_\psi < \varepsilon_{\psi_F}$ and (29) are satisfied, the following generically holds.*

The steady state is a sink for $0 < \varepsilon_v < \varepsilon_{v_H}$, undergoes a Hopf bifurcation at $\varepsilon_v = \varepsilon_{v_H}$, is a source for $\varepsilon_{v_H} < \varepsilon_v < \varepsilon_{v_T}$, undergoes a transcritical bifurcation at $\varepsilon_v = \varepsilon_{v_T}$, is a saddle for $\varepsilon_v > \varepsilon_{v_T}$.

Proposition 3 establishes that local indeterminacy and endogenous cycles occur under small increasing returns, a weak propensity to save and substitutable production factors.[25] In this respect, as it has initially been proved by Lloyd-Braga, Nourry and Venditti (2006), endogenous fluctuations arise under

[25] If $\epsilon_\psi > \epsilon_{\psi_F}$, local indeterminacy is not excluded. However, since increasing returns are required to be high, in contrast with empirical studies, we have omitted this case in Proposition 3.

mild conditions in such overlapping generations economies. More specifically, we observe that indeterminacy can occur if labor supply is sufficiently elastic ($\varepsilon_v < \varepsilon_{v_H}$).

4.2.2 Heterogeneity ($h > 0$)

Assume now that propensities to save are heterogeneous. By direct inspection of equations (18) and (19), we remark that D does not depend on h, whereas $\partial T/\partial h > 0$ if and only if $\varepsilon_v > \varepsilon_v^* \equiv \varepsilon_\psi/(1-\alpha)$.

The analysis simplifies under an additional mild restriction.

Assumption 6

$$\sigma < \frac{\alpha/s}{1-\alpha} + \frac{1}{\varepsilon_v^*}\left(\frac{\alpha/s}{1-\alpha} - 1\right) \tag{30}$$

Condition (30) is not too restrictive: The elasticity of capital-labor substitution is bounded from above by a value that is greater than 1 under Assumption 5. Inequality (30) entails that $\varepsilon_{v_H} < \varepsilon_v^*$, where, as before, ε_{v_H} is defined by $D = 1$ and does not depend on h.

In order to characterize local dynamics when heterogeneity matters, we need to know how the pair (T, D) moves in the plane when the bifurcation parameter ε_v varies in the range $(0, +\infty)$. The new locus, say Δ_2, is a branch of hyperbole (instead of a half-line) which depends on the degree of heterogeneity.

Let $\gamma \equiv \alpha(1-\sigma) - (1-\alpha+\varepsilon_\psi)(\sigma+\varepsilon_s-s)$. The system (18)-(19) gives implicitly D as a function of T, $D \equiv D(T)$. For simplicity, we define the inverse relation:

$$T(D) = \frac{D_1}{\alpha}\left[\gamma + \frac{\alpha D/D_1}{S(\sigma)} + \left(1-\alpha-\frac{\varepsilon_\psi}{D/D_1-1}\right)\frac{h}{s}\right], \text{ with } D > D_1 \tag{31}$$

We find easily that $D(T)$ is an increasing and convex function.[26]

In addition, we observe that $\lim_{\varepsilon_v \to 0}(T, D) = (-\infty, D_1)$, that is, a horizontal asymptote bounds from below Δ_2 on the left side. On the right side, $\lim_{\varepsilon_v \to +\infty}(T, D) = (+\infty, +\infty)$. Moreover Δ_2 crosses Δ_1 from above only once, exactly for $\varepsilon_v = \varepsilon_v^*$ (see Figure 5). This intersection point is invariant to h.

Two cases matters according to the location of Δ_2 with respect to B. We notice that Δ_2 goes through $B = (-2, 1)$, when

$$h_2 \equiv \frac{\gamma + [\sigma + 2\alpha + (1-\alpha)\varepsilon_s]/D_1}{\alpha\varepsilon_\psi/s - \sigma(\varepsilon_\psi - \varepsilon_{\psi_H})(1-\alpha)}\,\sigma s\left(\varepsilon_\psi - \varepsilon_{\psi_H}\right) \tag{32}$$

[26] Differentiating (31), we obtain:

$$\begin{aligned} T'(D) &= \frac{1}{S(\sigma)} + \frac{\varepsilon_\psi}{\alpha(D/D_1-1)^2}\frac{h}{s} > 0 \\ T''(D) &= -\frac{2\varepsilon_\psi}{\alpha D_1(D/D_1-1)^3}\frac{h}{s} < 0 \end{aligned}$$

and, finally, $D'(T) = 1/T'(D) > 0$ and $D''(T) = -T''(D)/[T'(D)]^3 > 0$.

where D_1 is given by (23).

When heterogeneity is moderate ($0 < h < h_2$), Δ_2 lies below B. By direct inspection of Figure 5, we deduce that Δ_2 is below the line (AB) for $0 < \varepsilon_v < \varepsilon_{v_F}$ (high elasticity of labor supply). When ε_v goes through ε_{v_F}, (T, D) crosses the line (AB). (T, D) lies inside the triangle (ABC) for $\varepsilon_{v_F} < \varepsilon_v < \varepsilon_{v_H}$. (T, D) crosses the segment $[BC]$ when ε_v goes through ε_{v_H}, and lies above (ABC) for $\varepsilon_{v_H} < \varepsilon_v < \varepsilon_{v_T}$. After crossing the line (AC) when ε_v goes through ε_{v_T}, eventually (T, D) lies on the right side of (AC) for $\varepsilon_v > \varepsilon_{v_T}$ (weak elasticity of labor supply).[27]

On the contrary, when heterogeneity is large ($h > h_2$), Δ_2 lies above B (see Figure 6). In this case, as ε_v moves from 0 to $+\infty$, Δ_2 starts on the left side of the line (AB) ($0 < \varepsilon_v < \varepsilon_{v_F}$), crosses (AB) above B ($\varepsilon_v = \varepsilon_{v_F}$), goes through the line (AC) ($\varepsilon_v = \varepsilon_{v_T}$) and definitely lies on the right-hand side of (AC) ($\varepsilon_v > \varepsilon_{v_T}$).

Conditions for indeterminacy and endogenous cycles are summarized in the following proposition.

Proposition 4 ***(Increasing returns)*** *If Assumptions 3, 5 and 6, and inequalities $\varepsilon_{\psi_H} < \varepsilon_\psi < \varepsilon_{\psi_F}$ and (29) are satisfied, the following generically holds.*

1. *Moderate heterogeneity ($0 < h < h_2$).*

   The steady state is a saddle for $0 < \varepsilon_v < \varepsilon_{v_F}$, undergoes a flip bifurcation at $\varepsilon_v = \varepsilon_{v_F}$, is a sink for $\varepsilon_{v_F} < \varepsilon_v < \varepsilon_{v_H}$, undergoes a Hopf bifurcation at $\varepsilon_v = \varepsilon_{v_H}$, is a source for $\varepsilon_{v_H} < \varepsilon_v < \varepsilon_{v_T}$, undergoes a transcritical bifurcation at $\varepsilon_v = \varepsilon_{v_T}$, is a saddle for $\varepsilon_v > \varepsilon_{v_T}$.

2. *Large heterogeneity ($h > h_2$).*

   The steady state is a saddle for $0 < \varepsilon_v < \varepsilon_{v_F}$, undergoes a flip bifurcation at $\varepsilon_v = \varepsilon_{v_F}$, is a source for $\varepsilon_{v_F} < \varepsilon_v < \varepsilon_{v_T}$, undergoes a transcritical bifurcation at $\varepsilon_v = \varepsilon_{v_T}$, is a saddle for $\varepsilon_v > \varepsilon_{v_T}$.

When a moderate degree of heterogeneity in the saving rates is introduced (Proposition 4, case 1), indeterminacy no longer emerges for a highly elastic labor supply. There is now a lower bound ε_{v_F} for values of ε_v compatible with indeterminacy. More precisely, indeterminacy arises if and only if the elasticity ε_v belongs to the interval $(\varepsilon_{v_F}, \varepsilon_{v_H})$. This interval shrinks with h, the degree of heterogeneity. Indeed ε_{v_H} does not depend on h, while ε_{v_F} increases.[28] In other terms, the range of elasticities of labor supply ($1/\varepsilon_v$) compatible with indeterminacy shrinks with h and heterogeneity in preferences stabilizes endogenous fluctuations.

[27] As above, ε_{v_F}, ε_{v_H} and ε_{v_T} are the values of the elasticity ε_v corresponding to the intersections of Δ_2 with (AB), $[BC]$ and (AC), respectively. For briefness, we omit the expression of ε_{v_T}, whereas ε_{v_H} is given by (28) and ε_{v_F} by (40) in the Appendix.

[28] For a proof of $\partial\varepsilon_{v_F}/\partial h > 0$, see the Appendix.

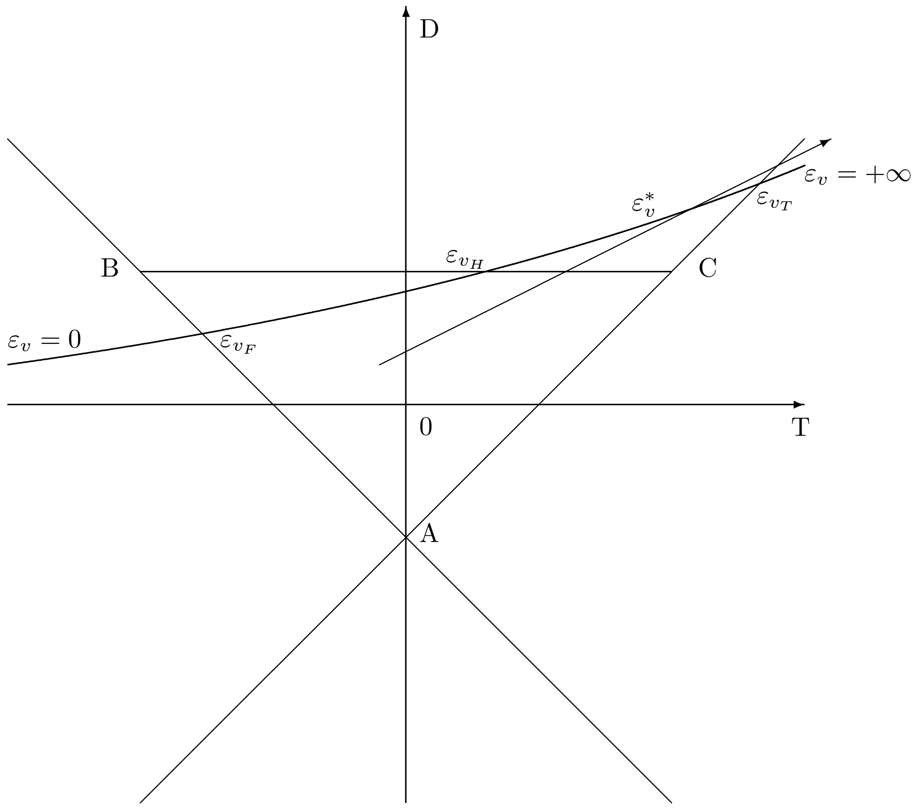

Figure 5: Increasing returns ($\varepsilon_\psi > 0$) and heterogeneity with $h > 0$ not too large

Furthermore, as under constant returns, indeterminacy is ruled out and the equilibrium becomes unique when heterogeneity is sufficiently large (Proposition 4, case 2).[29]

Therefore, we are allowed to conclude that, when consumers have heterogeneous preferences, indeterminacy requires more restrictive conditions and can be eventually eliminated. In other words, heterogeneity stabilizes expectations-driven fluctuations as it does under constant returns to scale, even if conditions for indeterminacy look like different.

[29]Notice that, since $h < 1/4$, the second case in Proposition 4 is of interest. In fact h_2 can be strictly less than $1/4$ and becomes weak when ε_ψ is sufficiently close to ε_{ψ_H} (see equation (32)).

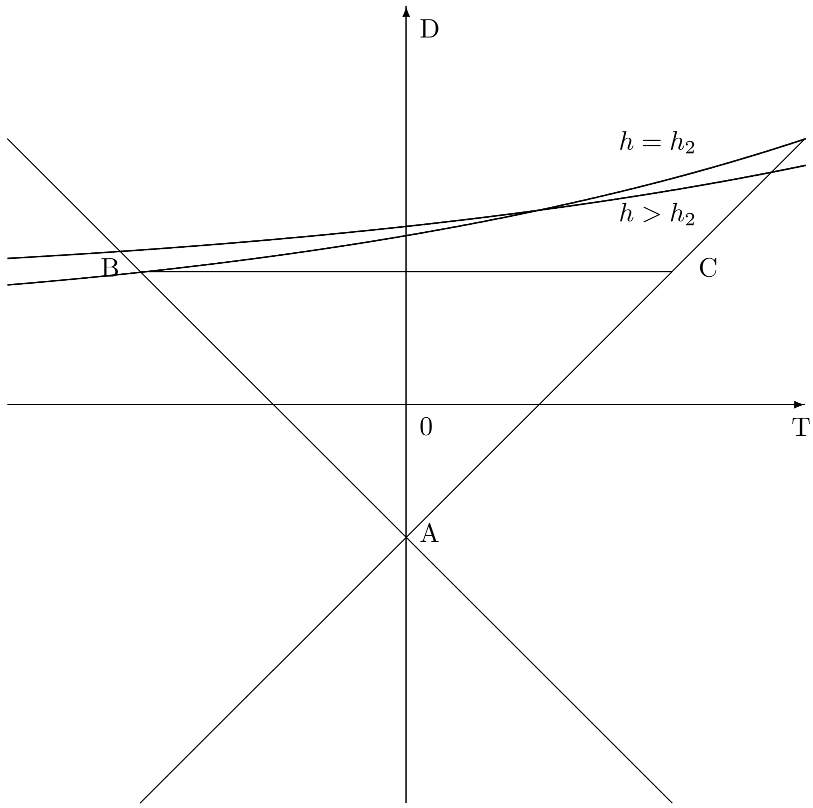

Figure 6: Increasing returns ($\varepsilon_\psi > 0$) and heterogeneity with $h > h_2$

5 Conclusion

A few papers have recently analyzed the role of heterogeneity between consumers in the stability properties of the path of capital accumulation, considering models with infinitely-lived agents. However, some authors have stressed the crucial role of the propensity to save on the determinacy properties of equilibria in overlapping generations economies with a representative consumer. This paper encompasses both views: we analyze the influence of heterogeneity in preferences, through different propensities to save, on the occurrence of endogenous fluctuations, considering an overlapping generations economy with capital accumulation, consumption in both periods and elastic labor supply.

Using a mean-preserving measure of dispersion, we show that under constant returns to scale, the introduction of heterogeneous propensities to save reduces the range of parameter values for which fluctuations due to self-fulfilling prophecies emerge. In particular, indeterminacy no longer occurs for a sufficiently elastic labor supply. One may conclude that heterogeneity stabilizes endogenous fluctuations since they appear for a narrower range of parameter values. Moreover, indeterminacy can even be ruled out, when heterogeneity becomes greater than a certain threshold. Introducing productive labor externalities, we prove the robustness of these results in the case of increasing returns to scale.

Our framework enables us to draw clear-cut conclusions about the influence of heterogeneity in preferences on indeterminacy. In contrast with some optimal growth models, where the planner's solution is used to study the stability effects of heterogeneity (Ghiglino (2005), Ghiglino and Venditti (2006)), we focus directly on the market regime and make some assumptions on preferences (homogeneity, separability) to derive standard parameters, like the propensity to save and the elasticity of labor supply.

An additional promising step could be to extend this line of research to market economies characterized by other forms of imperfections and analyze whether heterogeneity robustly reduces the indeterminacy range and can eliminate expectation-driven fluctuations.

6 Appendix

Existence of $s_i(r_{t+1})$. Equation (6) writes equivalently:

$$z_i\left(c_{i1t}/c_{i2t+1}\right) \equiv \frac{U_{i1}\left(c_{i1t}/c_{i2t+1},1\right)}{U_{i2}\left(1,c_{i2t+1}/c_{i1t}\right)} = r_{t+1}$$

where z_i is a strictly decreasing function. Then z_i is invertible and $c_{i1t} = z_i^{-1}\left(r_{t+1}\right)c_{i2t+1}$. Using the budget constraints (4) and (5), the saving rate $s_i\left(r_{t+1}\right)$ becomes: $s_i\left(r_{t+1}\right) = \left[1 + r_{t+1}z_i^{-1}\left(r_{t+1}\right)\right]^{-1}$.

Elasticity of U_i^*. Under Assumption 2, the Euler identity applies and, jointly with (6), gives[30]:

$$U_i\left(1-s_i, s_i r_{t+1}\right) = \left(1-s_i\right)U_{i1} + s_i U_{i2} r_{t+1} = U_{i1} \tag{33}$$

Using (33) and still (6), we have:

$$U_i^{*\prime}\left(r_{t+1}\right)r_{t+1} = \left[\left(U_{i2}r_{t+1} - U_{i1}\right)s_i' + U_{i2}s_i\right]r_{t+1} = s_i U_{i2} r_{t+1} = s_i U_{i1} = s_i U_i^*$$

$g(A)$ **is an increasing function.** The elasticity of g is computed from (10):

$$\begin{aligned}\frac{g'(A)\,A}{g(A)} &= 1 - \frac{\lambda_1 s_1\left(1-s_1\right)\left(1-\eta_1\right) + \lambda_2 s_2\left(1-s_2\right)\left(1-\eta_2\right)}{\lambda_1 s_1 + \lambda_2 s_2} \\ &> 1 - \frac{\lambda_1 s_1\left(1-s_1\right) + \lambda_2 s_2\left(1-s_2\right)}{\lambda_1 s_1 + \lambda_2 s_2} > 0\end{aligned}$$

since $\eta_i > 0$ for $i = 1, 2$.

[30] In the sequel, we drop the unnecessary arguments of the functions.

Determinant D and trace T of the Jacobian matrix J. Using (1), (2) and Assumption 1, we first compute the factor price elasticities. Denoting by r_i and w_i, $i \in \{k, l\}$, the derivatives of the real interest rate and the real wage with respect to k and l, we have:

$$\begin{bmatrix} kr_k/r & lr_l/r \\ kw_k/w & lw_l/w \end{bmatrix} = \begin{bmatrix} -(1-\alpha)/\sigma & (1-\alpha)/\sigma + \varepsilon_\psi \\ \alpha/\sigma & -\alpha/\sigma + \varepsilon_\psi \end{bmatrix} \tag{34}$$

With the notation $(l_{i1}, l_{i2}, l_{i3}, l_{i4}) \equiv (\partial l_i/\partial k_{t-1}, \partial l_i/\partial l_t, \partial l_i/\partial k_t, \partial l_i/\partial l_{t+1})$, the elasticities of labor supply ε_{ij}, $i = 1, 2$, $j = 1, \ldots, 4$, are defined and obtained from equations (13) as follows:

$$\begin{bmatrix} \varepsilon_{i1} & \varepsilon_{i2} \\ \varepsilon_{i3} & \varepsilon_{i4} \end{bmatrix} \equiv \begin{bmatrix} \frac{kl_{i1}}{l_i} & \frac{ll_{i2}}{l_i} \\ \frac{kl_{i3}}{l_i} & \frac{ll_{i4}}{l_i} \end{bmatrix} = \begin{bmatrix} \frac{1}{\varepsilon_{v_i}}\frac{\alpha}{\sigma} & -\frac{1}{\varepsilon_{v_i}}\left(\frac{\alpha}{\sigma} - \varepsilon_\psi\right) \\ -\frac{s_i}{\varepsilon_{v_i}}\frac{1-\alpha}{\sigma} & \frac{s_i}{\varepsilon_{v_i}}\left(\frac{1-\alpha}{\sigma} + \varepsilon_\psi\right) \end{bmatrix} \tag{35}$$

after using the elasticities of $U_i^*(r)$ and $v_i'(l_i)$. Finally, define:

$$\tilde{\varepsilon}_{i3} \equiv \varepsilon_{i3} - (1-s_i)(\eta_i - 1)(1-\alpha)/\sigma \tag{36}$$

$$\tilde{\varepsilon}_{i4} \equiv \varepsilon_{i4} + (1-s_i)(\eta_i - 1)\left[(1-\alpha)/\sigma + \varepsilon_\psi\right] \tag{37}$$

We linearize system (11)-(12) around the steady state $(k, l) = (1, 1)$ with $l_1 = l_2 = 1$ and we write the system in terms of elasticities (34). Equations (11) and (12) become, respectively:

$$\begin{aligned} &\left[1 - (\lambda_1 s_1 \tilde{\varepsilon}_{13} + \lambda_2 s_2 \tilde{\varepsilon}_{23}) w\right] \frac{dk_t}{k} - (\lambda_1 s_1 \tilde{\varepsilon}_{14} + \lambda_2 s_2 \tilde{\varepsilon}_{24}) w \frac{dl_{t+1}}{l} = [(\lambda_1 s_1 \varepsilon_{11} \\ &+ \lambda_2 s_2 \varepsilon_{21}) w + \alpha/\sigma] \frac{dk_{t-1}}{k} + \left[(\lambda_1 s_1 \varepsilon_{12} + \lambda_2 s_2 \varepsilon_{22}) w - (\alpha/\sigma - \varepsilon_\psi)\right] \frac{dl_t}{l} \end{aligned} \tag{38}$$

and

$$\begin{aligned} &-(\lambda_1 \varepsilon_{13} + \lambda_2 \varepsilon_{23}) \frac{dk_t}{k} - (\lambda_1 \varepsilon_{14} + \lambda_2 \varepsilon_{24}) \frac{dl_{t+1}}{l} \\ = \; &(\lambda_1 \varepsilon_{11} + \lambda_2 \varepsilon_{21}) \frac{dk_{t-1}}{k} + (\lambda_1 \varepsilon_{12} + \lambda_2 \varepsilon_{22} - 1) \frac{dl_t}{l} \end{aligned} \tag{39}$$

where $w = 1/s$ is the stationary wage. Define now $m_n \equiv \mu_1 s_1^n/\varepsilon_{v_1} + \mu_2 s_2^n/\varepsilon_{v_2}$, with $\mu_i \equiv \lambda_i s_i/(\lambda_1 s_1 + \lambda_2 s_2)$ and $n = -1, 0, 1$. Substituting (35), (36) and (37) in (38)-(39), we obtain the system $(dk_t/k, dl_{t+1}/l)^T = J\,(dk_{t-1}/k, dl_t/l)^T$, where:

$$J = \begin{bmatrix} \frac{1-\alpha}{\sigma} + \frac{1}{\varepsilon_s + m_1} & -\frac{1-\alpha}{\sigma} - \varepsilon_\psi \\ \frac{1-\alpha}{\sigma} & -\frac{1-\alpha}{\sigma} - \varepsilon_\psi \end{bmatrix}^{-1} \begin{bmatrix} \frac{\alpha}{\sigma}\frac{m_0+1}{m_1+\varepsilon_s} & \left(\varepsilon_\psi - \frac{\alpha}{\sigma}\right)\frac{m_0+1}{m_1+\varepsilon_s} \\ \frac{\alpha}{\sigma}\frac{m_{-1}}{m_0} & \left(\varepsilon_\psi - \frac{\alpha}{\sigma}\right)\frac{m_{-1}}{m_0} - \frac{1}{sm_0} \end{bmatrix}$$

is the Jacobian matrix. The determinant and the trace of this matrix are:

$$\begin{aligned} D &= \frac{1+m_0}{sm_0}\frac{\alpha}{1-\alpha+\sigma\varepsilon_\psi} \\ T &= D - \frac{m_{-1}}{m_0} \\ &+ \frac{\sigma + sm_{-1} - (m_0+1)(\alpha - sm_0\varepsilon_\psi) + (m_1 + \varepsilon_s)(1 - \alpha - sm_{-1}\varepsilon_\psi)}{sm_0(1-\alpha+\sigma\varepsilon_\psi)} \end{aligned}$$

In particular, according to Assumption 3, $\varepsilon_{v_1} = \varepsilon_{v_2} \equiv \varepsilon_v$ which implies $m_{-1} = s^{-1}/\varepsilon_v$, $m_0 = 1/\varepsilon_v$, $m_1 = (\mu_1 s_1 + \mu_2 s_2)/\varepsilon_v$. Using these expressions, we finally obtain:

$$\begin{aligned} D &= \frac{w\alpha(1+\varepsilon_v)}{1-\alpha+\sigma\varepsilon_\psi} \\ T &= D - w \\ &+ \frac{\varepsilon_\psi(1+1/\varepsilon_v) + w(1-\alpha+(\sigma-\alpha)\varepsilon_v + (m_1+\varepsilon_s)[(1-\alpha)\varepsilon_v - \varepsilon_\psi])}{1-\alpha+\sigma\varepsilon_\psi} \end{aligned}$$

or, equivalently, (18)-(19).

Proof of $\partial\varepsilon_{v_F}/\partial h > 0$. ε_{v_F} is defined by $1+T+D=0$. Let

$$\begin{aligned} a &\equiv \sigma+\alpha+\varepsilon_s(1-\alpha) > 0 \\ b &\equiv 2[\alpha+s(1-\alpha)] + \varepsilon_\psi[s-\varepsilon_s-\sigma(1-s)] \end{aligned}$$

Using (18) and (19), ε_{v_F} is solution of the following equation:

$$\varepsilon_v^2 a + \varepsilon_v[b+(1-\alpha)h/s] - \varepsilon_\psi h/s = 0$$

More explicitly,

$$\begin{aligned} \varepsilon_{v_F} &= \frac{-b-(1-\alpha)\frac{h}{s} + \sqrt{\left[b+(1-\alpha)\frac{h}{s}\right]^2 + 4a\varepsilon_\psi\frac{h}{s}}}{2a} \qquad (40) \\ \frac{\partial\varepsilon_{v_F}}{\partial h} &= \frac{1-\alpha}{2as}\left(\frac{b+(1-\alpha)\frac{h}{s}+\frac{2a\varepsilon_\psi}{1-\alpha}}{\sqrt{\left[b+(1-\alpha)\frac{h}{s}\right]^2+4a\varepsilon_\psi\frac{h}{s}}} - 1\right) \end{aligned}$$

We notice that $\partial\varepsilon_{v_F}/\partial h > 0$ iff

$$0 < b + \frac{a\varepsilon_\psi}{1-\alpha} < b + (1-\alpha)\frac{h}{s} + \frac{2a\varepsilon_\psi}{1-\alpha}$$

which is always true.

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